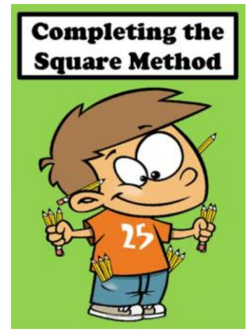
COMPLETING THE SQUARE



If you understand all the concepts in the chapter *Preparing for Completing the Square*, you're ready to tackle this chapter.

❑ *THE FIVE STEPS IN COMPLETING THE SQUARE*

We're now ready to combine many of our algebra skills into solving non-factorable quadratic equations, like $x^2 + 3x + 1 = 0$.

We start by assuming that the quadratic equation is in <u>standard form</u>. If it's not, we know we can always move things around to convert it to standard form:

$$ax^2 + bx + c = 0 \qquad \textbf{(where } \boldsymbol{a \neq 0}\textbf{)}$$

1. Make sure that the leading coefficient (the *a*) is 1. If it's not, divide each side of the equation (all terms) by *a*.
2. Move the constant to the other side of the equation.
3. Compute the "magic number" and add it to <u>both</u> sides of the equation. This step "*Completes the Square*."
4. Factor the left side, and then simplify the right side.
5. Solve the resulting equation by taking square roots, remembering that every positive number has two square roots (the Square Root Theorem).

<u>Note:</u> Steps 1 and 2 can be done in either order.

❑ EXAMPLES OF COMPLETING THE SQUARE

EXAMPLE 1: **Solve by Completing the Square:**

$$x^2 + 8x - 20 = 0$$

Solution: Even though this quadratic equation is factorable, we'll solve it by completing the square — a technique that will also work on problems that aren't factorable, as well as allow us to solve some problems with circles.

Step 1:

We must ensure that the leading coefficient (the a) is 1. It already is 1 ($x^2 = 1x^2$), so step 1 is done, and the equation remains the same:

$$x^2 + 8x - 20 = 0$$

Step 2:

Move the constant (the -20) to the right side of the equation by adding 20 to each side of the equation:

$$x^2 + 8x = 20$$

Step 3:

Now we calculate the *magic number*:

a) Calculate half of 8: $\frac{1}{2}(8) = 4$
b) Square that 4; the magic number is **16**.

Add the magic number to each side of the equation:

$$x^2 + 8x + \boxed{16} = 20 + \boxed{16}$$

Step 4:

Factor the left side and simplify the right side:

$$(x + 4)^2 = 36$$

Step 5:

Solve by taking square roots:

$$x + 4 = \pm\sqrt{36} \qquad \text{(36 has } \underline{\textbf{two}} \text{ square roots)}$$

$$x + 4 = \pm 6 \qquad \text{(simplify the radical)}$$

$$x = -4 \pm 6 \qquad \text{(subtract 4 from both sides)}$$

Using the plus sign $\Rightarrow$ $x = -4 + 6 = 2$

Using the minus sign $\Rightarrow$ $x = -4 - 6 = -10$

We have now solved our first equation by completing the square, and its solutions are

$$\boxed{x = 2 \text{ or } -10}$$

Note that we could have solved the quadratic equation by factoring. No matter the method, we would get the same solutions.

<u>EXAMPLE 2:</u> **Solve by Completing the Square:**

$$\mathbf{3x^2 - 5x + 1 = 0}$$

<u>Solution:</u> We proceed as we did above with the 5-step plan.

Step 1:

The first requirement for completing the square is a leading coefficient of 1. Since the leading coefficient in this problem is 3, we will have to divide both sides of the equation by 3:

$$\frac{3x^2 - 5x + 1}{3} = \frac{0}{3}$$

or, $\quad x^2 - \frac{5}{3}x + \frac{1}{3} = 0 \qquad$ (divide <u>all</u> terms by 3)

Step 2:

Move the constant to the right side, resulting in

$$x^2 - \frac{5}{3}x = -\frac{1}{3} \qquad \text{(subtract } \tfrac{1}{3} \text{ from both sides)}$$

Step 3:

It's now time for the magic number, the number that will complete the square. We calculate half of $-\frac{5}{3}$, square that result, and we'll have the "magic number" that will be added to each side of the equation.

Magic Number Calculation:

$$\frac{1}{2}\left(-\frac{5}{3}\right) = -\frac{5}{6}, \text{ and then } \left(-\frac{5}{6}\right)^2 = \mathbf{\frac{25}{36}}$$

Add this number to each side of the equation:

$$x^2 - \frac{5}{3}x + \boxed{\frac{25}{36}} = -\frac{1}{3} + \boxed{\frac{25}{36}}$$

Step 4:

Factoring the left side and adding the fractions on the right side gives us:

$$\left(x - \frac{5}{6}\right)^2 = \frac{13}{36} \qquad \left[-\frac{1}{3} + \frac{25}{36} = -\frac{12}{36} + \frac{25}{36} = \frac{13}{36}\right]$$

Step 5:

Now we take the square root of each side of the equation, remembering that the right side has **<u>two</u>** square roots:

$$x - \frac{5}{6} = \pm\sqrt{\frac{13}{36}}$$

Next, we isolate the x by adding $\frac{5}{6}$ to each side of the equation:

$$x = \frac{5}{6} \pm \sqrt{\frac{13}{36}}$$

Split the radical:

$$x = \frac{5}{6} \pm \frac{\sqrt{13}}{\sqrt{36}} \qquad \left[\text{from the rule: } \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}\right]$$

And simplify the bottom radical:

$$x = \frac{5}{6} \pm \frac{\sqrt{13}}{6}$$

Combining the fractions into a single fraction (the LCD is 6) produces the final answer:

$$\boxed{x = \frac{5 \pm \sqrt{13}}{6}}$$

Our quadratic has two solutions.

Homework

1. At what point in Completing the Square could you determine that the quadratic equation you're trying to solve has NO solution?

2. Solve each quadratic equation by Completing the Square:

a. $y^2 - 6y + 5 = 0$

b. $x^2 + 13x + 30 = 0$

c. $z^2 + 5z - 14 = 0$

d. $2n^2 - n - 1 = 0$

e. $12t^2 - 5t - 3 = 0$

f. $10a^2 + 7a + 1 = 0$

g. $x^2 + 25 = 10x$

h. $4u^2 + 20u + 25 = 0$

i. $w^2 = -w - 5$

j. $2h^2 + 1 = h$

3. Solve each quadratic equation by Completing the Square:

a. $x^2 + 3x + 1 = 0$

b. $y^2 - 4y + 2 = 0$

c. $2a^2 + 6a + 3 = 0$

d. $n^2 + 8n - 2 = 0$

e. $3u^2 - 4u - 2 = 0$

f. $t^2 + 10t + 3 = 0$

g. $5w^2 + w + 1 = 0$

h. $2x^2 = -5x - 1$

i. $g^2 = 3g - 5$

j. $3m^2 = 1 - 4m$

4. Solve each quadratic equation by Completing the Square:

a. $q^2 + 8q + 15 = 0$

b. $2x^2 - 7x + 1 = 0$

c. $3n^2 + n - 5 = 0$

d. $4a^2 - 5a = -3$

e. $5w^2 = -2 - 7w$

f. $z^2 + 1 = 2z$

Solutions

1. As soon as you notice that you've reached the square root of a negative number, you can stop and conclude that the quadratic equation has no solution (in the real numbers).

2.

a. $y = 1, 5$

b. $x = -3, -10$

c. $z = 2, -7$

d. $n = 1, -\frac{1}{2}$

e. $t = \frac{3}{4}, -\frac{1}{3}$

f. $a = -\frac{1}{2}, -\frac{1}{5}$

g. $x = 5$

h. $u = -\frac{5}{2}$

i. No solution

j. No solution

3. a. $x = \frac{-3 \pm \sqrt{5}}{2}$ b. $y = 2 \pm \sqrt{2}$ c. $a = \frac{-3 \pm \sqrt{3}}{2}$

d. $n = -4 \pm 3\sqrt{2}$ e. $u = \frac{2 \pm \sqrt{10}}{3}$ f. $t = -5 \pm \sqrt{22}$

g. No solution h. $x = \frac{-5 \pm \sqrt{17}}{4}$ i. No solution

j. $m = \frac{-2 \pm \sqrt{7}}{3}$

4. a. $q = -5, -3$ b. $x = \frac{7 \pm \sqrt{41}}{4}$ c. $n = \frac{-1 \pm \sqrt{61}}{6}$

d. No solution e. $w = -\frac{2}{5}, -1$ f. $z = 1$

“The true sign of intelligence
is not knowledge,
but *imagination*.”

Albert Einstein